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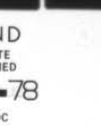UNCLASSIFIED   USCEE-496   AFOSR-TR-78-0196   NL

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Volume 4

RECURSIVE DERIVATION OF REFLECTION COEFFICIENTS
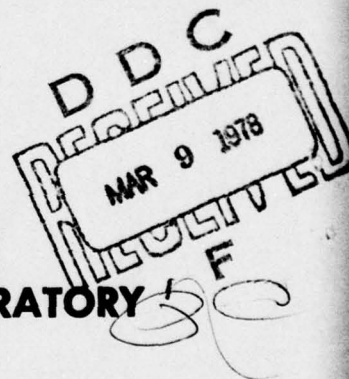FROM NOISY SEISMIC DATA

N. E. Nahi, J. M. Mendel, and L. M. Silverman

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Interim Technical Report

July 1977

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Volume 2.   "Single-Channel White Noise Estimators for Predictive Deconvolution," J. M. Mendel, USCEE Report 488, January 1976.

Volume 3.   "A Time-Domain Approach to Seismogram Synthesis for Layered Media," N. E. Nahi and J. M. Mendel, USCEE Report 490, June 1976.

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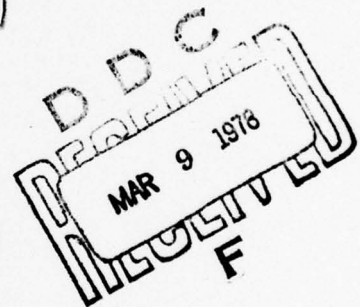University of Southern California

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ABSTRACT

We consider plane-wave motion at normal incidence in a horizontally layered system. The system is assumed lossless, and only the compressional waves are treated. A procedure is introduced for determining the reflection coefficients of the layered system when the observed seismic data may contain random noise. No deconvolution of the measured seismic data is required by the procedure when the input is a narrow wavelet.

1. INTRODUCTION

In recent years much attention has been given to the problem of determining reflection coefficients for a layered media from the observed seismic data [1-4]. In line with the customary assumptions and restrictions, we also limit our attention to a horizontally stratified nonabsorptive earth with vertically traveling plane compressional waves. Such a system is completely described by a set of reflection coefficients and travel times within layers.

A fundamental procedure described in detail in the above references for deriving values of the reflection coefficients can be summarized by the following assumptions and steps.

Standard Assumptions:

    (A1)   The input wavelet is assumed known.

    (A2)   The data is assumed noise free.

    (A3)   The layered system is assumed to have uniform travel times between layers where a number of the layers are hypothetical, i.e., they may not correspond to an actual interface of the substructure and are associated with zero reflection and unity transmission coefficients.

Standard Steps:

    (S1)   The observed seismic data is deconvolved using the input waveform to produce the system response to a unit spike input.

    (S2)   The number of layers is chosen high enough to result in travel times short compared with the inverse of the bandwidth of the observed seismic data.

    (S3)   The deconvolved data is sampled with sampling interval

equal to the chosen one-way travel time between layers.[*]

(S4)   The system structure is used to arrive at a set of normal equations (linear simultaneous equations) in terms of reflection coefficients and the discretized and deconvolved observed data.

(S5)   The normal equations of the preceding step have the Toeplitz structure which makes it possible to utilize the very efficient Levinson algorithm to recursively solve for the reflection coefficients.

In this paper the method of solution to the inverse problem stated above is fundamentally modified to cope with the existence of the noise in the measurement data, often without need for any deconvolution. More specifically, although again a uniform layered system is assumed, the choice of number of layers can now be made independent of the sampling rate requirement of the data (step (S2) above) often resulting in the need for far fewer layers. No deconvolution is necessary (step (S1)) for wavelets of duration of the order of twice the layer travel times. The exact deconvolution of step (S1) is either not possible in practice or, at the least, will further aggrevate the harmful effects of the noise in the observation [5]. Furthermore, the deconvolution is a time consuming operation. Finally, the procedure is very simple to derive and does not need the concepts of z-transforms, minimum phase, forward and backward polynomials, spectral factorization, etc. The results reduce to the existing solution of the inverse problem in the absence of noise and with a spike input signal (wavelet) [1].

---

[*]Of course the deconvolution may be performed in discrete time using the same sampling interval.

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2. STATEMENT OF THE PROBLEM

We are considering a uniform K layered system and normal incident compressional waves. Figure 1 represents such a system where $d_j(t)$ is the down-going wave at the bottom of the jth layer and $u_j(t)$ is the up-going wave at the top of the layer. The reflection, downward transmission and up-ward transmission coefficients associated with the interface at the bottom of jth layer are denoted r_j, t_j and t'_j respectively where $t_j = 1 + r_j$, $t'_j = 1 - r_j$. The one way travel time between layers is denoted by τ.
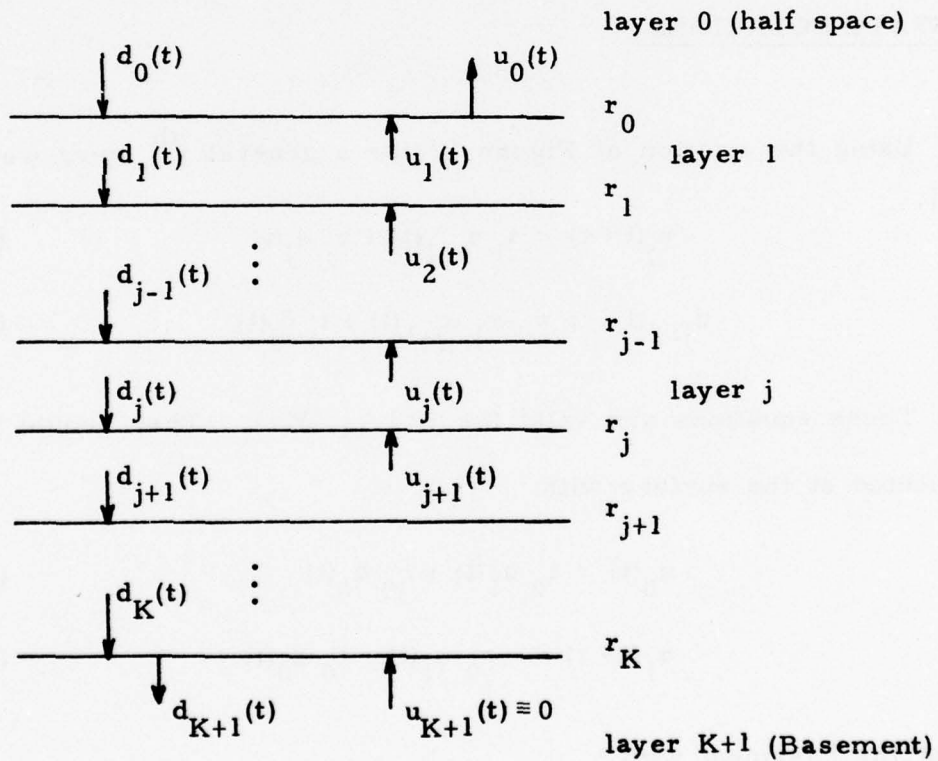

Figure 1. K Layered System

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The input to the system, $d_0(t)$, is assumed known (the wavelet) and the output may be either $u_1(t)$ (in the marine environment) or $u_0(t)$. The measured seismic data, $y(t)$, consists of the output and an additive noise component $n(t)$. The source of this noise may be the instrument measurement noise, the uncertainty in the knowledge of the input wavelet or response to unwanted inputs (ambient noise). It is desired to process $y(t)$, $t \geq 0$ and derive values for the reflection coefficients r_j, $j = 0, 1, \ldots, K$ (r_0 may be assumed known in cases such as the marine environment).

3. STATE EQUATIONS

Using the notation of Figure 1, for a general j^{th} layer we have [6, 7].

$$u_j(t + \tau) = t_j' u_{j+1}(t) + r_j d_j(t) \qquad (1)$$

$$d_{j+1}(t + \tau) = -r_j u_{j+1}(t) + t_j d_j(t) \qquad (2)$$

These equations are valid for $j = 1, \ldots, K-1$. They should be augmented at the surface with

$$u_0(t) = t_0' u_1(t) + r_0 d_0(t) \qquad (3)$$

$$d_1(t + \tau) = -r_0 u_1(t) + t_0 d_0(t) \qquad (4)$$

and at the basement with

$$u_K(t + \tau) = t_K' u_{K+1}(t) + r_K d_K(t) = r_K d_K(t) \qquad (5)$$

$$d_{K+1}(t) = -r_{K-1} u_{K+1}(t) + t_K d_K(t) = t_K d_K(t) \qquad (6)$$

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Equations (3,4) and (5,6) can be derived from (1) and (2) [letting $j = 0, 1, \ldots, K$] by noting that $u_0(t)$ is taken at the <u>bottom</u> of layer 0 and $d_{K+1}(t)$ at the top of layer-(K+1) and $u_{K+1}(t) = 0$. The term $d_{K+1}(t)$ represents the down-going wave leaving the last interface and is not reflected by any other interface; hence $u_{K+1}(t) = 0$. These equations, called causal functional, are not difference equations since t is the continuous time variable [6].

4. <u>A GENERALIZED ENERGY TRANSFER (KUNETZ) RELATION</u>

Consider ε to be a non-negative continuous or discrete variable with dimension of time. Equations (1) and (2) [where $j = 0, 1, \ldots, K$] are multiplied by $u_j(t + \tau + \varepsilon)$ and $d_{j+1}(t + \tau + \varepsilon)$ respectively resulting in

$$u_j(t + \tau) \, u_j(t + \tau + \varepsilon) = t_j^2 \, u_{j+1}(t) \, u_{j+1}(t + \varepsilon) + r_j^2 \, d_j(t) \, d_j(t + \varepsilon)$$
$$+ \, r_j \, t_j'[u_{j+1}(t) \, d_j(t + \varepsilon) + u_{j+1}(t + \varepsilon) \, d_j(t)] \qquad (7)$$

$$d_{j+1}(t + \tau) \, d_{j+1}(t + \tau + \varepsilon) = r_j^2 \, u_{j+1}(t) \, u_{j+1}(t + \varepsilon) + t_j^2 \, d_j(t) \, d_j(t + \varepsilon)$$
$$- \, r_j \, t_j[u_{j+1}(t) \, d_j(t + \varepsilon) + u_{j+1}(t + \varepsilon) \, d_j(t)] \qquad (8)$$

Multiplying (7) by t_j/t_j' and adding the resulting expression to (8) yields

$$d_{j+1}(t + \tau) \, d_{j+1}(t + \tau + \varepsilon) + (t_j/t_j') \, u_j(t + \tau) \, u_j(t + \tau + \varepsilon)$$
$$= (t_j/t_j') \, d_j(t) \, d_j(t + \varepsilon) + u_{j+1}(t) \, u_{j+1}(t + \varepsilon) \qquad (9)$$

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Let us define the following correlation-type functions

$$D_j(\varepsilon) \overset{\Delta}{=} \int_{-\infty}^{+\infty} d_j(t)\, d_j(t+\varepsilon)\, dt \tag{10}$$

$$U_j(\varepsilon) \overset{\Delta}{=} \int_{-\infty}^{+\infty} u_j(t)\, u_j(t+\varepsilon)\, dt \tag{11}$$

Integrating both sides of Eq. (9) from $-\infty$ to $+\infty$, and using Eqs. (10) and (11), we find that

$$D_{j+1}(\varepsilon) - U_{j+1}(\varepsilon) = t_j/t_j'[D_j(\varepsilon) - U_j(\varepsilon)] \tag{12}$$

where $j = 0, 1, 2, \ldots, K$. This is a generalization of the well-known [1] energy transfer (Kunetz) relation. Note that in our derivation, input $d_0(t)$ is not assumed to be an impulse and the seismic data is not discretized. An abstract generalization of (12) is given in Appendix A.

Iterating (12), starting with $j = \ell$ and ending with $j = K$, we obtain

$$D_{K+1}(\varepsilon) = \frac{\prod\limits_{\ell}^{K} t_i}{\prod\limits_{\ell}^{K} t_i'} [D_\ell(\varepsilon) - U_\ell(\varepsilon)] \tag{13}$$

where ℓ can take on the values of $0, 1, \ldots,$ or K. In the marine case this relationship is used with $\ell = 1$. In the non-marine case (Appendix B), it is used with $\ell = 0$.

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5. APPLICATION TO MARINE ENVIRONMENT

In this section we will direct our attention to the marine case and shall express $D_{K+1}(\varepsilon)$ in terms of measured signals. To do this, we set $r_0 = 1$ and we see from (13) that we must express $D_1(\varepsilon) - U_1(\varepsilon)$ in terms of the measured signals. The first layer can be depicted as in Figure 2.
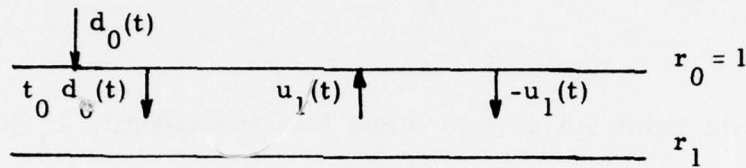
Figure 2. First layer in the marine case.

Observe that (4) becomes

$$d_1(t + \tau) = -u_1(t) + 2d_0(t) . \tag{14}$$

From (10), (11), and (14), we can evaluate the difference term $D_1(\varepsilon) - U_1(\varepsilon)$,

$$D_1(\varepsilon) - U_1(\varepsilon) \triangleq P(\varepsilon) = \int_{-\infty}^{+\infty} \{[2d_0(t) - u_1(t)][2d_0(t + \varepsilon) - u_1(t + \varepsilon)]$$
$$- u_1(t)\, u_1(t + \varepsilon)\}\, dt \tag{15}*$$

or

$$P(\varepsilon) = \int_{-\infty}^{+\infty} 4d_0(t)\, d_0(t + \varepsilon)\, dt - \int_{-\infty}^{+\infty} 2d_0(t)\, u_1(t + \varepsilon)\, dt$$

$$- \int_{-\infty}^{+\infty} 2u_1(t)\, d_0(t + \varepsilon)\, dt ; \tag{16}$$

---

*Because of the range of integration in (10), we can also express $D_j(\varepsilon)$ as $D_j(\varepsilon) = \int_{-\infty}^{\cdot} d_j(t + \tau)\, d_j(t + \tau + \varepsilon)\, dt$. We use this form of (10) in our development of $D_1(\varepsilon) - U_1(\varepsilon)$.

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hence, $P(\varepsilon)$ can be evaluated from a knowledge of $d_0(t)$ and $u_1(t)$ for any desired ε. Observe, also, that (13) with $\ell = 1$ can be written in terms of $P(\varepsilon)$, using (15), as

$$D_{K+1}(\varepsilon) = \frac{\prod\limits_{1}^{K} t_i}{\prod\limits_{1}^{K} t_i'} \, P(\varepsilon) \qquad\qquad (17)$$

We should point out at this stage that the quantity $u_1(t)$ needed in (16) is only available through the observation

$$y(t) = u_1(t) + n(t)$$

where $n(t)$ is the additive noise. Consequently, $P(\varepsilon)$ is not physically available; however, we can define $\overline{P}(\varepsilon)$ by replacing $y(t)$ for $u_1(t)$ in (16),

$$\overline{P}(\varepsilon) \triangleq \int_{-\infty}^{+\infty} 4 \, d_0(t) \, d_0(t+\varepsilon) \, dt - \int_{-\infty}^{+\infty} 2 \, d_0(t) \, y(t+\varepsilon) \, dt$$

$$ \qquad\qquad (18)$$

$$ - \int_{-\infty}^{+\infty} 2 \, y(t) \, d_0(t+\varepsilon) \, dt$$

which can also be written as

$$\overline{P}(\varepsilon) = P(\varepsilon) + N(\varepsilon) \quad , \qquad\qquad (19)$$

where

$$N(\varepsilon) \triangleq - \int_{-\infty}^{+\infty} 2 \, d_0(t) \, n(t+\varepsilon) \, dt - \int_{-\infty}^{+\infty} 2 \, n(t) \, d_0(t+\varepsilon) \, dt \qquad . \qquad (20)$$

The statistics of noise term $N(\varepsilon)$ can be determined in terms of those of $n(t)$. Using $\overline{P}(\varepsilon)$ in place of $P(\varepsilon)$ in (17) yields

$$D_{K+1}(\varepsilon) = \frac{\prod\limits_{1}^{K} t_i}{\prod\limits_{1}^{K} t_i'} \; \bar{P}(\varepsilon) \qquad (21)$$

where $\bar{P}(\varepsilon)$ is a known quantity. Equation (21) is a fundamental relationship which will be used in the derivation of the inverse procedure. A discussion of the non-marine case is given in Appendix B.

6. DERIVATION OF THE NORMAL EQUATIONS

The following property is basic to our derivation of the <u>normal</u> <u>equations.</u>

Structural Property:

The function $d_{K+1}(t)$ satisfies the equation

$$d_{K+1}[t+K\tau] + a_1 \, d_{K+1}[t+(K-2)\tau] + \dots + a_{K-1} \, d_{K+1}[t-(K-2)\tau]$$

$$+ r_0 \, r_K \, d_{K+1}[t-K\tau] = \prod\limits_{0}^{K} t_i \, d_0(t) \qquad (22)$$

Note that the coefficient of the highest term of the left hand side is unity and that of the lowest term is $r_0 r_K$. The precise form of the other coefficients is not important. A proof of this result is given in Appendix C.

Let us multiply both sides of (22) by $d_{K+1}[t+(K-2i)\tau]$ and integrate from $-\infty$ to ∞ for $i = 0, 1, \dots, K$.

This results in $K+1$ simultaneous equations, which, using (10), become

$$
\begin{bmatrix}
D_{K+1}(0) & D_{K+1}(2\tau) & D_{K+1}(4\tau) & \cdots & D_{K+1}(2K\tau) \\
D_{K+1}(2\tau) & D_{K+1}(0) & D_{K+1}(2\tau) & \cdots & \\
D_{K+1}(4\tau) & D_{K+1}(2\tau) & D_{K+1}(0) & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D_{K+1}(2K\tau) & \cdot & \cdot & \cdot & D_{K+1}(0)
\end{bmatrix}
\begin{bmatrix}
1 \\
a_1 \\
. \\
. \\
. \\
r_K
\end{bmatrix}
= 2\prod_1^K t_i
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
. \\
. \\
. \\
\alpha_{K+1}
\end{bmatrix}
$$

$$(23)$$

where we have substituted $t_0 = 1 + r_0 = 2$ to represent the marine environment and

$$
\alpha_{i+1} = \int_{-\infty}^{+\infty} d_0(t)\, d_{K+1}(t + K\tau - 2i\tau)\, dt, \quad i = 0, 1, 2, \ldots, K \quad (23a)
$$

Substituting for $D_{K+1}(\varepsilon)$ from (21), we find that (23) reduces to

$$
\begin{bmatrix}
\overline{P}(0) & \overline{P}(2\tau) & \overline{P}(4\tau) & \cdots & \overline{P}(2K\tau) \\
\overline{P}(2\tau) & \overline{P}(0) & \overline{P}(2\tau) & \cdots & \\
\overline{P}(4\tau) & \overline{P}(2\tau) & \overline{P}(0) & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\overline{P}(2K\tau) & & \cdots & & \overline{P}(0)
\end{bmatrix}
\begin{bmatrix}
1 \\
a_1 \\
. \\
. \\
. \\
r_k
\end{bmatrix}
= 2\prod_1^K t_i'
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
. \\
. \\
. \\
\alpha_{K+1}
\end{bmatrix}
\quad (24)
$$

Note that the $(K+1) \times (K+1)$ matrix on the left has the Toeplitz structure. The terms $N(0), N(2\tau), \ldots$ which appear in $\overline{P}(0), \overline{P}(2\tau), \ldots$, are random variables with known statistics; they will be zero if the seismic data is noise free (i.e., $n(t) \equiv 0$). Observe, also, that the

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first and last elements of the vector on the left-hand side of (24) are unity and r_K, respectively, by virtue of the property which we just stated for the d_{K+1} causal functional equation.

Equation (24) provides the starting point for our procedure for identifying the reflection coefficients. Note that in general the α_i are functions of $d_{K+1}(t)$, a signal which is not determinable; however, we will show in the following section that when the input wavelet is narrow enough (not necessarily a spike), (24) has a unique solution for the reflection coefficients in terms of observable data.

7. SPECIAL CASE OF NARROW WAVELET

Let us now consider the case where $d_0(t)$ does not extend* beyond 2τ, i.e.

$$d_0(t) = 0 \qquad t < 0, \; t > 2\tau \; . \qquad (25)$$

Since the time of arrival at the Kth interface is $K\tau$ and the time of arrival of the first reflections is $(K+2)\tau$,

$$d_{K+1}(t) = 0 \qquad\qquad t < K\tau \qquad\qquad (26a)$$

$$= 2 \prod_1^K t_i \, d_0(t - K\tau) \qquad K\tau < t \leq (K+2)\tau \qquad (26b)$$

$$= \text{more complicated terms} \quad t > (K+2)\tau \qquad (26c)$$

---

* If this condition is not satisfied, we can always deconvolve the data to achieve this. Since the requirement here is not to deconvolve down to an impulse function (only (25) has to be satisfied), this results in a more practical solution.

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From (23a), (26a), (26b), and (25) we see that

$$\alpha_1 = \int_{-\infty}^{+\infty} d_0(t)\, d_{K+1}(t+K\tau)\, dt = 2\prod_1^K t_i \int_0^{2\tau} d_0^2(t)\, dt \ , \tag{27}$$

and that

$$\alpha_{i+1} = \int_{-\infty}^{+\infty} d_0(t)\, d_{K+1}\left[t+K\tau - 2i\tau\right] dt = 0 \qquad i=1,\ldots,K \tag{28}$$

Note now that (24) will have precisely $K+1$ unknowns, K of them in the vector multiplying the Toeplitz matrix and one on the right-hand side, α_1.

Finally, Normal Equation (24) can be written in a compact matrix form, as

$$\overline{P}_K\, a_K = C_K \tag{29}$$

where $\overline{P}_K$ is a $(K+1) \times (K+1)$ Toeplitz matrix with the first row being[*] $[\overline{P}(0), \overline{P}(2\tau), \ldots, \overline{P}(2K\tau)]$; a_K is a $K+1$ column vector with first and last elements 1 and r_K, respectively; and, $C_K = \mathrm{col}\,(\beta_K, 0, 0, \ldots, 0)$ and

$$\beta_K = 2\prod_{i=1}^K (1 - r_i^2) \int_0^{2\tau} d_0^2(t)\, dt$$

The Normal Equation (29) can be solved for a_K. This only produces one of the K reflection coefficients, namely r_K. We will show, in the following, that in the case of the marine environment, nested within (29) are a set of normal equations, the solutions of which produce each one of the reflection coefficients. The absence of this useful property in the non-marine case renders the procedure of this paper inapplicable

----

[*]For a narrow wavelet and $\varepsilon \geq 2\tau$, the calculation of $\overline{P}(\varepsilon)$ simplifies since (18) reduces to
$$\overline{P}(\varepsilon) = -2 \int_0^{2\tau} d_0(t)\, y(t+\varepsilon)\, dt \ .$$

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in that case [see Appendix B].

Let us now hypothesize a j-layer system (i.e., the basement layer is the j-th) consisting of the top j-layers of the above K-layer system $(K \geq j)$. Clearly, from (29), we have

$$\bar{P}_j a_j = C_j \qquad\qquad (30)$$

where a_j will again have 1 and r_j as first and last elements. We shall now show that, in the case of the marine environment, $\bar{P}_j$ is a $(j+1) \times (j+1)$ Toeplitz matrix composed of the top left corner of $\bar{P}_K$; i.e., its first row is given by $[\bar{P}(0), \bar{P}(2\tau), \ldots, \bar{P}(2j\tau)]$.

For the moment let us ignore the additive noise term in (19). Let us denote by $u_1^j(t)$ the response of the j-layer system (i.e., the term $u_1(t)$ in Figure 2 is replaced by $u_1^j(t)$). In (16), due to the fact that $d_0(t) = 0$ for $t > 2\tau$, the last value of $u_1(t)$ contributing to $P(\epsilon)$ is $u_1(2\tau + \epsilon)$. In determination of $\bar{P}_j$, with elements $\bar{P}(\epsilon)$, $\epsilon = 0, \ldots, 2j\tau$, for a j-layered system, therefore, the last value of $u_1^j(t)$ contributing to $\bar{P}_j$ is $u_1^j[2(j+1)\tau]$. On the other hand, $u_1(t)$ is the response of the K-layer system, and

$$u_1(t) = u_1^j(t) \qquad\qquad 0 \leq t \leq 2(j+1)\tau \qquad (31)$$

since the first return from the interfaces below the j-th will not appear earlier than $t = 2(j+1)\tau$. Hence, the elements of $\bar{P}_j$, which are functions of $u_1^j(t)$ and $\bar{P}_K$ which are functions of $u_1(t)$, are identical when (31) is satisfied. In other words, the numerical values of $\bar{P}(0), \ldots, \bar{P}(2j\tau)$ will be identical to those of the K-layer system for all $j \leq K$. Furthermore, the additive noise term in (19) is independent of the number of layers, as is evident from (20).

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The set of normal equations given by (30), for $j = 1, \ldots, K$, can now be solved for the vectors a_j and, hence, their last elements, r_j, $j = 1, \ldots, K$. The matrix $\overline{P}_K$ is Toeplitz and consequently, the Levinson algorithm [1] can be used to solve for the vectors a_j, $j = 1, \ldots, K$ recursively.

Since the r_j's are reflection coefficients, for the solution to this problem to be acceptable, each r_j must be less than unity in magnitude. It is shown in Appendix D that any solution of (30) with $\beta_j > 0$ for all j yields a set of r_j's which satisfy this condition. Moreover if P_K is positive definite, a compatible solution with $\beta_j > 0$ is guaranteed. We see therefore, that the requirement that $|r_j| < 1$ has nothing to do with a specific method of solution of the normal equations (i. e. the Levinson procedure). This result is different from the comparable result in [1]-[3], where one is left with the impression that a specific method of solution leads to $|r_j| < 1$.

8. REMARKS

A comparison between the procedure of this paper and the standard procedures described in [1]-[4] is warranted. Let us consider the noise term n(t) to be white. Clearly, (20) indicates that the random variables $N(\cdot)$ have finite variances. For this case (n(t) white), had we performed the necessary deconvolution and sampling required by the classical approach to the inverse problem, the resulting $N(\cdot)$ random variables would have infinite variance, clearly rendering the approach meaningless.[*] Of course, "approximate" deconvolution will

*If n(t) is not wide-band, then the variance of $N(\cdot)$ may not be infinite, but will be very large.

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eliminate this problem but at a great sacrifice in the information

available within the seismic data. It should also be noted that, for the

narrow wavelets, no deconvolution is required by the procedure

outlined in this paper.

Finally, the procedure of this paper can be applied (see Appendix

E) to the classical solution of the inverse problem as it appears in

[1]-[4].

9. CONCLUSIONS

We have developed a procedure for extracting reflection

coefficients from noisy data which we feel is a substantial generalization

of similar procedures which have been reported in the literature.

Associated with these earlier procedures are *Standard Assumptions*

and Steps (see Introduction, page 1) which include requirements that

the data be noise free and that the observed seismic data be deconvolved.

The procedure of our paper avoids these restrictive requirements.

Furthermore, our procedure totally avoids the concepts of z-transforms,

minimum phase, spectral factorization, forward and reverse polynomial

manipulations, etc., which appear in the literature on this subject.

Finally, since our derivation is so straightforward, it suggests a

number of extensions, including the following, which are presently

under study: (1) nonstandard locations of source and sensors (e.g.,

both in the first layer); (2) minimum mean-square estimation of the

reflection coefficients; and (3) optimal prefiltering of noisy data

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(exploiting the potential utility of the abstract Kunetz relation in
Appendix A).

ACKNOWLEDGEMENT

APPENDIX A

An Abstract Kunetz-Type Relationship

A generalized Kunetz relationship was derived in Section 4 by starting with the following state equations,

$$u_j(t + \tau) = t_j' u_{j+1}(t) + r_j d_j(t) \qquad (A1)$$

$$d_{j+1}(t + \tau) = -r_j u_{j+1}(t) + t_j d_j(t) \qquad (A2)$$

and performing the following operations: (a) time shift $u_j(t + \tau)$ and $d_{j+1}(t + \tau)$ by ε to obtain $u_j(t + \tau + \varepsilon)$ and $d_{j+1}(t + \tau + \varepsilon)$ [this is done by operating on both sides of (A1) and (A2) by a linear advance operator]; (b) multiply $u_j(t + \tau)$ by $u_j(t + \tau + \varepsilon)$, and $d_j(t + \tau)$ by $d_j(t + \tau + \varepsilon)$; (c) add $(t_j/t_j') u_j(t + \tau) u_j(t + \tau + \varepsilon)$ to $d_j(t + \tau) d_j(t + \tau + \varepsilon)$; (d) integrate the resulting expression from $-\infty$ to $+\infty$; and (e) iterate the integrated expression across the K-layers.

Step (c) is a pivotal one; for, it leads to a cancellation of terms common to both $(t_j/t_j') u_j(t + \tau) u_j(t + \tau + \varepsilon)$ and $d_j(t + \tau) d_j(t + \tau + \varepsilon)$. Step (e) is also important in that it permits us to use the boundary condition, that $u_{K+1}(t) = 0$.

Steps (a), (b), and (d) involve specific operations. We show here that these operations can be abstracted (i.e., generalized), and, that we can obtain an abstract Kunetz-type relationship which is valid for much more general operations than shift, multiply, and integrate.

Let L_j denote a linear operator for the jth layer. Operate on both sides of (A1) and (A2) with L_j, to show that

$$L_j u_j(t+\tau) = t'_j L_j u_{j+1}(t) + r_j L_j d_j(t) \qquad (A3)$$

$$L_j d_{j+1}(t+\tau) = -r_j L_j u_{j+1}(t) + t_j L_j d_j(t) \qquad (A4)$$

Let $\circ$ denote an operation that satisfies certain group properties (e.g. $\circ$ could be multiplication, convolution, etc.); then,

$$
\begin{aligned}
\left[L_j u_j(t+\tau) \right] \circ u_j(t+\tau) = \; & t'^2_j \left[L_j u_{j+1}(t) \right] \circ u_{j+1}(t) \\
& + t'_j r_j \left[L_j u_{j+1}(t) \right] \circ d_j(t) \\
& + t'_j r_j \left[L_j d_j(t) \right] \circ u_{j+1}(t) \\
& + r^2_j \left[L_j d_j(t) \right] \circ d_j(t) \qquad (A5)
\end{aligned}
$$

and

$$
\begin{aligned}
\left[L_j d_{j+1}(t+\tau) \right] \circ d_{j+1}(t+\tau) = \; & r^2_j \left[L_j u_{j+1}(t) \right] \circ u_{j+1}(t) \\
& - t_j r_j \left[L_j u_{j+1}(t) \right] \circ d_j(t) \\
& - t_j r_j \left[L_j d_j(t) \right] \circ u_{j+1}(t) \\
& + t^2_j \left[L_j d_j(t) \right] \circ d_j(t) \qquad (A6)
\end{aligned}
$$

From (A5) and (A6), we find that

$$\left[L_j d_{j+1}(t+\tau) \right] \circ d_{j+1}(t+\tau) + (t_j/t'_j) \left[L_j u_j(t+\tau) \right] \circ u_j(t+\tau)$$

$$= \left[L_j u_{j+1}(t) \right] \circ u_{j+1}(t) + (t_j/t'_j) \left[L_j d_j(t) \right] \circ d_j(t) \qquad (A7)$$

which can also be written, as

$$\left[L_j d_{j+1}(t+\tau) \right] \circ d_{j+1}(t+\tau) - \left[L_j u_{j+1}(t) \right] \circ u_{j+1}(t)$$

$$= (t_j/t'_j) \left\{ \left[L_j d_j(t) \right] \circ d_j(t) - \left[L_j u_j(t+\tau) \right] \circ u_j(t+\tau) \right\} \qquad (A8)$$

Let N denote another linear operator. Operate on both sides of (A8) with N, to show that

$$N\{[L_j\, d_{j+1}(t+\tau)]\circ d_{j+1}(t+\tau)\} - N\{[L_j\, u_{j+1}(t)]\circ u_{j+1}(t)\}$$

$$= (t_j/t_j')\{N\{[L_j\, d_j(t)]\circ d_j(t)\} - N\{[L_j\, u_j(t+\tau)]\circ u_j(t+\tau)\}\} \qquad (A9)$$

Now, iterate (A9) backwards, starting with $j = K$, for which we know that $u_{K+1}(t) = 0$, to see that

$$N\{[L_K\, d_{K+1}(t+\tau)]\circ d_{K+1}(t+\tau)\} = \frac{\displaystyle\prod_{i=1}^{K} t_i}{\displaystyle\prod_{i=1}^{K} t_i'}\, \{N\{[L_1\, d_1(t)]\circ d_1(t)\}$$

$$- N\{[L_1\, u_1(t+\tau)]\circ u_1(t+\tau)\}\,\}$$

$$(A10)$$

Equation (A9) is an abstract <u>Kunetz-types relationship</u>. It reduces to Eq. (12) when

$$L_j = z_\varepsilon^{-1}, \text{ all } j, \text{ where } z_\varepsilon^{-1} \text{ is an } \varepsilon \text{ sec.}$$
advance operator.

$$\circ = x \qquad \text{(multiplication)}$$
and
$$N = \int_{-\infty}^{\infty} (\cdot)\, dt$$

$$\left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\}\qquad (A11)$$

Equation (A10) is then equivalent to Eq. (13).

The applicability of the abstract Kunetz-type relationship using other operations [e.g., L_j = a low pass filter, $\circ = *$ (convolution), N = another filtering operation] remains to be studied. What is

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interesting, though, is the fact that a very general relationship – (A9) or (A10) – exists for a layered media system, and, that this relationship does not, in general, have anything to do with energy or an energy spectrum.

APPENDIX B

Non-Marine Case

We question what limits the results of this paper mainly to the marine problem. The answer is two fold. First, although here a structural property similar to (22) is still valid leading to (29), the extension of (29) to (30) is only possible for the marine case, since, in the non-marine case, Eq. (14) is $d_1(t+\tau) = -r_0 u_1(t) + t_0 d_0(t)$, which means that the cancellation of $u_1(t) u_1(t+\epsilon)$ terms in (15) will not occur. Therefore, $P(\epsilon)$ will be a function of the <u>entire</u> $u_1(t)$ for $t \epsilon [0, \infty)$, regardless of how narrow $d_0(t)$ is. This suggests that, for the non-marine case, it seems we can only (at least directly) solve for the last reflection coefficient (r_K) and not r_j, $j = 1, \ldots, K-1$. Second, as we shall show below, the nature of the noise term, $N(\epsilon)$, is more complex in the non-marine case, to the point that its statistics cannot be obtained from those of the observation noise alone.

Let

$$P_0(\epsilon) \triangleq D_0(\epsilon) - U_0(\epsilon) \qquad (B1)$$

From (10) and (11), we then have

$$P_0(\epsilon) = \int_{-\infty}^{+\infty} d_0(t)\, d_0(t+\epsilon)\, dt - \int_{-\infty}^{+\infty} u_0(t)\, u_0(t+\epsilon)\, dt\,; \qquad (B2)$$

and, from (13),

$$D_{K+1}(\epsilon) = \frac{\prod_0^K t_i}{\prod_0^K t_i'}\, P_0(\epsilon) \qquad (B3)$$

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The measurement in the non-marine case is given by

$$y(t) = u_0(t) + n(t) \qquad\qquad (B4)$$

Since $u_0(t)$ is not available, we may use $y(t)$ in its place by defining $\overline{P}_0(\varepsilon)$, as

$$\overline{P}_0(\varepsilon) \triangleq \int_{-\infty}^{+\infty} d_0(t)\, d_0(t+\varepsilon)\, dt \ - \ \int_{-\infty}^{+\infty} y(t)\, y(t+\varepsilon)\, dt \qquad\qquad (B5)$$

which, using (B2), can be written as

$$\overline{P}_0(\varepsilon) = P_0(\varepsilon) + N(\varepsilon) \qquad\qquad (B6)$$

where

$$N(\varepsilon) = -\int_{-\infty}^{+\infty} n(t)\, u_0(t+\varepsilon)\, dt \ - \ \int_{-\infty}^{+\infty} u_0(t)\, n(t+\varepsilon)\, dt$$

$$\qquad\qquad (B7)$$

$$-\int_{-\infty}^{+\infty} n(t)\, n(t+\varepsilon)\, dt$$

Finally, (B3) becomes

$$D_{K+1}(\varepsilon) = \frac{\displaystyle\prod_0^K t_i}{\displaystyle\prod_0^K t_i'}\ \overline{P}_0(\varepsilon) \qquad\qquad (B8)$$

Note that here the statistics of $N(\varepsilon)$ depend on the statistics of $n(t)$ <u>and</u> the availability of $u_0(t)$; however, $u_0(t)$ is not available and can only be obtained through a noisy measurement. Additionally, the third term in $N(\varepsilon)$ is nonlinear in $n(t)$. Observe, from (20), that no such nonlinear noise term occurs in the marine case.

APPENDIX C

Proof of Structural Property

In the following, a constructive proof of Eq. (22) is given.

Equations (1) and (2) can be re-written as

$$u_j(t) = \frac{1}{t_j} [u_{j+1}(t - \tau) + r_j d_{j+1}(t)] \qquad (C1)$$

and

$$d_j(t) = \frac{1}{t_j} [r_j u_{j+1}(t) + d_{j+1}(t + \tau)] \quad , \qquad (C2)$$

where (C1) is obtained by eliminating the term $d_j(t)$ between (1) and (2). For $j = 0$, (C2) becomes

$$d_0(t) = \frac{1}{t_0} [r_0 u_1(t) + d_1(t + \tau)] \qquad (C3)$$

Replacing for $u_1(t)$ and $d_1(t + \tau)$ from (C1) and (C2) yields

$$d_0(t) = \frac{1}{t_0 t_1} \{ r_0 u_2(t - \tau) + r_1 u_2(t + \tau) + r_0 r_1 d_2(t) + d_2(t + 2\tau)\} \qquad (C4)$$

Replacing for the terms in the right of (C4) from (C1) and (C2) yields

$$d_0(t) = \frac{1}{t_0 t_1 t_2} \{ r_0 u_3(t - 2\tau) + (r_1 + r_1 r_2) u_3(t) + r_2 u_3(t + 2\tau)$$

$$+ r_0 r_2 d_3(t - \tau) + (r_1 + r_1 r_2) d_3(t + \tau) + d_3(t + 3\tau)\} \qquad (C5)$$

Repeating the process another time yields

$$d_0(t) = \frac{1}{t_0 t_1 t_2 t_3} \{ r_0 u_4(t - 3\tau) + (r_1 + r_1 r_2 + r_0 r_2 r_3) u_4(t - \tau)$$

$$+ (r_2 + r_1 + r_1 r_2) u_4(t + \tau) + r_3 u_4(t + 3\tau) + r_0 r_3 d_4(t - 2\tau)$$

$$+ (r_1 + r_1 r_2 + r_0 r_2) d_4(t) + (r_1 + r_1 r_2 + r_2 r_3) d_4(t + 2\tau)$$

$$+ d_4(t + 4\tau) \qquad (C6)$$

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Equation (C3) is for a one layer system where $d_1(t)$ is defined at the bottom of layer 1 (see Figure 1). That equation can also be used to describe a zero layer system (i.e., two semi-infinite half spaces) by setting $u_1(t) = 0$ and moving d_1 from the bottom of layer 1 to the top of that layer. This is accomplished by setting $t = t - \tau$ in the d_1 term. The resulting equation is

$$t_0 \, d_0(t) = d_1(t) \tag{C7}$$

We proceed similarly for (C4), (C5), and (C6). For example, (C6) is for a four layer system where $d_4(t)$ is defined at the bottom of layer 4. That equation can also be used to describe a three layer system by setting $u_4(t) = 0$ and moving d_4 from the bottom of layer 4 to the top of that layer. This is accomplished by setting $t = t - \tau$ in all d_4 terms. The resulting equation is

$$\prod_0^3 t_i \, d_0(t) = d_4(t+3\tau) + a_1 d_4(t+2\tau) + a_2 d_4(t-\tau) + r_0 r_3 \, d_4(t-3\tau) \tag{C8}$$

This development demonstrates the existence of our equation (22). Set K=0 in (22) to obtain (C7), and set K = 3 in (22) to obtain (C8).

APPENDIX D

Some Properties of Toeplitz Matrices

Consider a set of a equations of the type encountered in Section 7:

$$A_i x_i = b_i , \qquad i = 1, \ldots, n \qquad (D1)$$

where the A_i, $i = 1, \ldots, n$ are the leading principal minors of the symmetric Toeplitz matrix $A = A_n$, and b_i and x_i are $i \times 1$ vectors with the special form

$$b_i = \begin{bmatrix} K_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad , \qquad x_i = \begin{bmatrix} 1 \\ \\ \\ y_i \end{bmatrix} = \begin{bmatrix} 1 \\ y_i^1 \\ \vdots \\ y_i^{i-1} \end{bmatrix}$$

with $b_1 = K_1$, $x_1 = 1$ (in the notation of Section 7, $A_i = \overline{P}_i$, $x_i = a_i$, $b_i = C_i$, $K_i = \beta_i$ and $y_i^{i-1} = r_i$). Several properties of equations of the type (D1) which bear on the solution of equation (30) will now be derived.

First note that if A is Toeplitz, then it can be partitioned as

$$A_i = \begin{bmatrix} A_1 & m_i' \\ m_i & A_{i-1} \end{bmatrix}$$

so that (D1) is equivalent to the pair of equations

$$A_1 + m_i' y_i = K_i \qquad (D2)$$

$$m_i + A_{i-1} y_i = 0 \qquad (D3)$$

Also, if $\det A_{i-1} \neq 0$, then by a well known determinant identity [8]

$$\det A_i = \det A_{i-1} (A_1 - m_i' A_{i-1}^{-1} m_i) \quad .$$

Furthermore, if $\det A_{i-1} \neq 0$ (D3) can be solved for y_i and substituted into (D2) yielding

$$K_i = A_1 - m_i' A_{i-1}^{-1} m_i$$

Hence if $\det A_{i-1} \neq 0$ we have the identity

$$K_i = \frac{\det A_i}{\det A_{i-1}} \qquad (D4)$$

<u>Theorem</u>: A symmetric Toeplitz matrix A_n is positive definite if and only if for each $i = 1, \ldots, n$, the equation (D1) has a unique solution y_i for some $K_i > 0$.

<u>Proof</u>: We prove sufficiency by induction on the sequence of matrices A_i. For $i = 1$ (D1) merely implies $A_1 = a_{11} = K_1 > 0$. Suppose now we have shown that $A_i > 0$, $i = 1, \ldots, k-1$. Applying (D2) for $i = k$ and the fact that $\det A_{k-1} > 0$ by the induction hypothesis we have by (D4) that

$$\det A_k = K_k \det A_{k-1} > 0$$

which completes the induction step since $A_i > 0$ is i^{th} leading principle minor of A_k.

To prove necessity note that if $A > 0$ then $\det A_i > 0$ for $i = 1, \ldots, n$ so that, for each i, the equations (D2) and (D3) have a unique solution with

$$K_i = \frac{\det A_i}{\det A_{i-1}} > 0$$

Corollary: If A is a symmetric Toeplitz matrix with leading principal minors A_i, $i = 1, \ldots, n$ and $y_i' = [y_i^1 \ldots y_i^{i-1}]$ is a solution of (D1) for $K_i > 0$, then $|y_i^{i-1}| < 1$ for $i = 1, \ldots, n$.

Proof: By our theorem, $A_i > 0$ so that $Q_i = A_i^{-1} > 0$. Hence

$$[10 \ldots 0 - 1] Q_i \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} = (q_{11} + q_{ii}) - (q_{i1} + q_{1i}) > 0$$

Since A_i is symmetric, Q_i is symmetric so that $q_{i1} = q_{1i}$. Further, since A_i is Toeplitz

$$A_i = \begin{bmatrix} A_1 & m \\ m' & A_{i-1} \end{bmatrix} = \begin{bmatrix} A_{i-1} & \ell \\ \ell' & A_1 \end{bmatrix}$$

so that

$$q_{11} = \frac{\det A_{i-1}}{\det A_i} = q_{ii} \tag{D5}$$

Hence, $2(q_{11} - q_{i1}) > 0$, or

$$q_{i1} < q_{11}$$

If y_i satisfies (D1) with $K_i > 0$, then the last element of y_i satisfies

$$y_i^{i-1} = q_{i1} K_i < q_{11} K_i = 1$$

since by (D4) and (D5), $q_{11} = 1/K_i$.

By a similar argument we can show that $q_{11} + q_{i1} > 0$ which implies $q_{i1} > q_{11}$, so that

$$y_i^{i-1} = q_{i1} K_i > -q_{11} K_i = -1 \quad .$$

APPENDIX E

The Classical Discrete-Time Solution to Inverse Problem

It may be of some interest to relate the results of this paper to those of reference [1]-[4].

Let the variable t assume discrete values which are a multiple of τ; i.e.,

$$t = k\tau \quad , \quad k = 0, 1, \ldots \tag{E1}$$

Furthermore, let the input be a unit pulse (unit spike) at $t = 0$

$$d_0(t) = \Delta(0) \tag{E2}$$

With this notation, the only change to be made in the derivation is the redefinition of quantities $D_j(\epsilon)$ and $U_j(\epsilon)$ in (10) and (11). Let

$$\epsilon = i\tau \tag{E3}$$

$$D_j(k\tau) \overset{\Delta}{=} \sum_{i=-\infty}^{+\infty} d_j[k\tau] \, d_j[(k+i)\tau] \tag{E4}$$

$$U_j(k\tau) \overset{\Delta}{=} \sum_{i=-\infty}^{+\infty} u_j[k\tau] \, u_j[(k+i)\tau] \tag{E5}$$

The energy transfer relation (13) remains the same, and (16) is replaced by

$$P(i\tau) = 4 \, \Delta(i\tau) - 2 \, u_1(i\tau) \tag{E6}$$

We also disregard the presence of the noise term in (19). Equation (22) is again valid for $d_0(t) \implies \Delta_0(t)$, $t = 0$.

The remainder of the derivation follows in similar manner resulting in the following normal equation (to replace (29)):

- 29 -

$$
\begin{bmatrix} P(0) & P(2\tau) & \cdots \\ P(2\tau) & P(0) & \\ \vdots & & \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ \vdots \\ r_k \end{bmatrix} = 2 \prod_{1}^{K} (1 - r_i^2) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\, E7 \,)
$$

which is the result found in Refs. [1]-[4].

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